Fundamentals of Engineering (FE) Exam Mathematics Review

Dr. Garey Fox
Professor and Buchanan Endowed Chair
Biosystems and Agricultural Engineering
October 16, 2014

Reference Material from FE Review
Instructor’s Manual, Prepared by Gregg C. Wagener, PE, Professional Publications, Inc
Straight Line

- General form of straight line:
  \[ Ax + By + C = 0 \]

**NOTE:** Highest exponent for any variable is 1
Straight Line

- Standard form (slope-intercept form):
  \[ y = mx + b \]
- Point-slope form:
  \[ y - y_1 = m(x - x_1) \]
- Equation for the slope:
  \[ m = \frac{y_2 - y_1}{x_2 - x_1} \]
- Distance between two points:
  \[ d = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2} \]
Intersecting Straight Lines

• Angle between the lines: \( \alpha = \tan^{-1} \left( \frac{m_2 - m_1}{1 + m_2 m_1} \right) \)

• If lines are perpendicular: \( m_1 = -\frac{1}{m_2} \)
Example Problem

What is the slope of the line $2y = 2x + 4$?

(A) 1    (B) 1/2    (C) -2    (D) Infinite

Converting to general form, $2y - 2x - 4 = 0$

$A = -2$, $B = 2$

$m = -A/B = -(-2/2) = 1$

Therefore (A) is correct.
Example Problem

What is the distance between the point \( y = 0, x = -3 \) and the intercept of the line \( y + 4x - 4 = 0 \) and the y axis?

(A) -5  (B) 2  (C) 4  (D) 5

The line intercepts the y axis at \((0,4)\)

\[ d = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2} \]

\[ \sqrt{(4 - 0)^2 + (0 - (-3))^2} = 5 \]

Therefore (D) is correct.
Algebra

- Solving two linear equations simultaneously
  - First, look for a simple substitution
  - Second, look for a simple reduction

What is the solution to the simultaneous equations $2x + 4y = 5; \ 3x + y = 5$

- (A) $x = 1.5; \ y = 0.5$
- (B) $x = 0.5; \ y = 1.5$
- (C) $x = -1.5; \ y = 0.5$
- (D) $x = 1.5; \ y = -0$

Multiply 2nd equation by 4, subtract, solve for $x = 1.5$, substitute and solve for $y = 0.5$
Therefore (A) is correct.
Quadratic Equation

- Any equation of the form...

\[ ax^2 + bx + c = 0 \]

- Roots of the equation \((x_1, x_2)\):

\[ x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]
Quadratic Equation

• Discriminant determines the roots...

\[ b^2 - 4ac > 0 \quad \text{Real and Unequal} \]
\[ b^2 - 4ac = 0 \quad \text{Real and Equal} \]
\[ b^2 - 4ac < 0 \quad \text{Complex and Unequal} \]
Example

What are the roots of the equation $50 \, x^2 + 5(x - 2)^2 = -1$, where $x$ is a real valued variable?

(A) $-6.12$ and $-3.88$  
(B) $-0.52$ and $0.70$  
(C) $7.55$  
(D) no real solution

Convert to standard form

$50 \, x^2 + 5(x^2 - 4x + 4) = -1$

$55 \, x^2 - 20x + 21 = 0$

Roots=$\frac{-20 \pm \sqrt{(20)^2 - (4)(55)(21)}}{(2)(55)}$

The quantity under the radical is negative. Therefore (D) is correct.
Cubic Equation

• Any equation of the form...

\[ ax^3 + bx^2 + cx + d = 0 \]

• Roots of the equation \((x_1, x_2, x_3)\):
  – Simplify and find easiest roots
  – Look for answers that can be eliminated
  – Plug and chug!
Example

What are the roots of the cubic equation $x^3 - 8x - 3 = 0$?

(A) $-7.9, -3, -0.38$
(B) $-3, -2, 2$
(C) $-3, -0.38, 2$
(D) $-2.6, -0.38, 3$

With inspection we see that $+3$ is a root. This eliminates (A), (B), and (C) as possibilities.
Conic Sections

- Any of several curves produced by passing a plane through a cone.
Conic Sections

• Two Angles:

\[ \theta = \text{angle between the vertical axis and the cutting plane} \]
\[ \phi = \text{cone-generating angle} \]

• Eccentricity, e, of a conic section:

\[ e = \frac{\cos \theta}{\cos \phi} \]
Conic Sections

- Quadratic Equation:
  \[ Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0 \]

- If \( A = C = 0 \), then conic section = line
  - If \( A = C \neq 0 \), then conic section = circle
  - If \( A \neq C \):
    - \( B^2-AC<0 \), then conic section = ellipse
    - \( B^2-AC>0 \), then conic section = hyperbola
    - \( B^2-AC=0 \), then conic section = parabola
Example

What geometric shape is represented by the formula 
\[(y - 2)(y + 2) = 14x - 12?\]

(A) Parabola  
(B) Ellipse  
(C) Hyperbola  
(D) Circle

Convert to the general form:
\[y^2 - 4 - 14x +12 = 0\]
\[0x^2 + (2)(0)xy + 1y^2 + (2)(7)x + (2)(0)y + 8 = 0\]
\[B^2 - AC = 0 - 0 = 0\]
Therefore (A) is correct.

\[Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0\]
Parabola

parabola $\theta = \phi$

$e = 1$

$y = k$

vertex $(h, k)$

focus $(h + \frac{p}{2}, k)$

latus rectum $x = h + \frac{p}{2}$

directrix $x = h - \frac{p}{2}$

parabolic axis
Parabola

- For Center (vertex) at $(h,k)$, focus at $(h+p/2, k)$, directrix at $x=h-p/2$ and that opens horizontally
  
  $$(y - k)^2 = 2p(x - h)$$

- Opens to Right if $p>0$
- Opens to Left if $p<0$
What is the focus of the parabola \( y^2 - 4y - 16x = -36 \)?

(A) (36,16)  (B) (16,36)  (C) (2, -2)  (D) (6,2)

\[
y^2 - 4y + 4 = 16(x - 2) \\
(y - 2)^2 = (2)(8)(x-2) \\
\text{Focus at } (h + p/2, k) = (2 + 8/2, 2) = (6,2) \\
\text{Therefore (D) is correct.}
Ellipse

Ellipse \((\phi < \theta < 90^\circ)\)

\[0 < e < 1\]
Ellipse

- For Center (vertex) at (h,k), semimajor distance (a) and semiminor distance (b)

\[
\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1
\]

\[e = c / a\]
Circle

- Special ellipse

\[(x - h)^2 + (y - k)^2 = r^2\]
Circle

- Length, $t$, to a circle from a point $(x', y')$:

$$t = (x' - h)^2 + (y' - k)^2 - r^2$$
Example

What is the center of the circle \( x^2 - 6x + (y - 4)^2 = 0 \)?

(A) (6, -4)  (B) (4,6)  (C) (4,3)  (D) (3,4)

Convert to the standard form by adding 9 to both sides

\[
x^2 - 6x + 9 + (y - 4)^2 = 9
\]

\[
(x - 3)(x - 3) + (y - 4) = 9
\]

\[
(x - 3)^2 + (y - 4)^2 = 3^2
\]

The center is at (3,4)

Therefore (D) is correct.
Hyperbola

(d) hyperbolas \(0 \leq \theta < \phi\)

\[ e > 1 \]
Hyperbola

• For Center (vertex) at \((h,k)\) and opening horizontally

\[
\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1
\]

\[e = c/a, \quad e > 1\]
Three-Dimensional Objects

• Sphere centered at \((h, k, m)\) with radius \(r\):

\[
(x - h)^2 + (y - k)^2 + (z - m)^2 = r^2
\]

• Distance between two points in 3-d space:

\[
d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}
\]
Mensuration

• Mensuration (measurements) of perimeter, area, and other geometric properties

• Handbook for Formulas!
Example

An elliptical jogging track is laid out with 40 m from the center on the major axis and 30 m from the center on the minor axis. What answer is nearest to how many laps would have to be run to go a kilometer?

(A) 13  (B) 6.4  (C) 6.3  (D) 4.5

\[ p = 2\pi \left( \frac{a^2 + b^2}{2} \right)^{1/2} = 2\pi \left( \frac{40^2 + 30^2}{2} \right)^{1/2} \]

\[ = 222 \text{ m/lap} \]

Laps = 1000 m/222 m/lap = 4.5

Therefore (D) is correct.
Logarithms

• Think of logarithms as exponents...

\[ b^c = x \]

- Exponent is \( c \) and expression above is the logarithm of \( x \) to the base \( b \)

\[ \log_b(x) = c \Rightarrow b^c = x \]

- Base for common logs is 10 (\( \log = \log_{10} \))
- Base for natural logs is \( e \) (\( \ln = \log_e \)), \( e = 2.71828 \)

• Identities - HANDBOOK!
Trigonometry

- sin, cos, tan
- cot = 1/tan, csc = 1/sin, sec=1/cos
- Law of sines and cosines!
- Identities - HANDBOOK!
Trigonometry

• Plug in sin and cos for all tan, cot, csc, and sec
• Simplify and look for a simple identity OR work backwards by simplifying the possible answers
Example

What is the equivalent expression for \( \sin 2\alpha \)?

(A) \( \frac{1}{2} \sin \alpha \cos \alpha \)  (B) \( 2 \sin \alpha \cos(\frac{1}{2} \alpha) \)  (C) \( -2 \sin \alpha \)  (D) \( \frac{2 \sin \alpha}{\sec \alpha} \)

Plugging in \( \cos \alpha = 1/\sec \alpha \) into (D) gives: \( 2 \sin \alpha \cos \alpha \)

\( 2 \sin \alpha = 2 \sin \alpha \cos \alpha \) is a valid identity.

Therefore (D) is correct.
Complex Numbers

- Combination of real and imaginary numbers (square root of a negative number)
  \[ i = \sqrt{-1} \]

- Rectangular Form: \( a + ib \)
Complex Numbers

• Identities - HANDBOOK!
  – Algebra is done separately for real and imaginary parts!
  – Multiplying:
    • Rectangular Form: Note that $i^2 = -1$

• Polar Coordinates: Converting $z = a + ib$ to $z = r(\cos \theta + i \sin \theta)$
  – HANDBOOK!
  – Multiplication: Magnitude multiply/divide, Phase angle add/subtract
Complex Numbers

- Another notation for polar coordinates: $z = re^{i\theta}$ (Euler’s Identity...HANDBOOK!)
- Convert Rectangular/Polar - HANDBOOK!
- Roots - the kth root, $w$, of a complex number $z = r(\cos \theta + i \sin \theta)$ is given by:

$$w = k^{\frac{1}{r}} \left[ \cos \left( \frac{\theta}{k} + n \frac{360^\circ}{k} \right) + i \sin \left( \frac{\theta}{k} + n \frac{360^\circ}{k} \right) \right]$$
Example

What is the sum of the complex numbers \((4 + i7)\) and \((6 + i9)\)?

(A) 10 + i16  
(B) i11 + i15  
(C) i 26  
(D) 10 + i17

\[(4 + i7) + (6 + i9) = (4 + 6) + i(7 + 9) = 10 + i16\]

Therefore (A) is correct.
Example

What is the product of the complex numbers $2 - i2$ and $\sqrt[4]{32e^{\frac{i\pi}{4}}}$?

(A) 16 \hspace{1cm} (B) $i16$ \hspace{1cm} (C) $16e^{\frac{i\pi}{4}}$ \hspace{1cm} (D) $16(1 - i)$

$2 - i2$ has a magnitude of $\sqrt{8}$ and an associated angle of $-45^\circ$, so $2 - i2 = \sqrt{8}e^{-\frac{i\pi}{4}}$.

$\sqrt[4]{32e^{\frac{i\pi}{4}}} \sqrt[4]{8e^{-\frac{i\pi}{4}}} = \sqrt[4]{32\sqrt[4]{8e^{\frac{i\pi}{4}}}} = 16$

Therefore (A) is correct.
Matrices

- $m \times n = \text{number of rows } \times \text{number of columns}$
- Square Matrix: $m=n$ (order)
- Multiplication:
  - Two matrices: $A = m \times n$, $B = n \times s$
    
    $AB = m \times s$
    
    $BA = \text{Not Possible}$
Matrices

• Multiplication

\[
\begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6
\end{bmatrix}
\begin{bmatrix}
7 & 10 \\
8 & 11 \\
9 & 12
\end{bmatrix}
= 
\begin{bmatrix}
(1 \times 7) + (2 \times 8) + (3 \times 9) & 68 \\
122 & 167
\end{bmatrix}
\]

• Addition: only possible if matrices have same number of rows and columns

\[
\begin{bmatrix}
1 & 2 \\
3 & 4
\end{bmatrix}
+ 
\begin{bmatrix}
1 & 2 \\
3 & 4
\end{bmatrix}
= 
\begin{bmatrix}
2 & 4 \\
6 & 8
\end{bmatrix}
\]
Matrices

• Identity Matrix:

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

• Transpose of a \( m \times n \) matrix is \( n \times m \) matrix constructed by taking \( i^{\text{th}} \) row and making it the \( i^{\text{th}} \) column

\[
\begin{bmatrix}
1 & 6 & 9 \\
5 & 4 & 2 \\
7 & 3 & 8
\end{bmatrix}^T = \begin{bmatrix}
1 & 5 & 7 \\
6 & 4 & 3 \\
9 & 2 & 8
\end{bmatrix}
\]
Matrices

• Determinants: Formulas in HANDBOOK!

\[
\begin{vmatrix}
1 & 2 \\
3 & 4 \\
\end{vmatrix}
= 1 \times 4 - 3 \times 2 = -2
\]

• Minor of element \(a_{i,j}\) = determinant when row \(i\) and \(j\) are crossed out (if \(i+j\) is even, then multiply the determinant by 1 and if odd, then multiply the determinant by -1)
Matrices

- **Cofactor Matrix** = minor for all elements of the original matrix with appropriate sign

\[
\begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 8
\end{pmatrix}
\]

cofactor of 1 is

\[
\begin{pmatrix}
5 & 6 \\
8 & 8
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 8
\end{pmatrix}
\text{Cofactor} =
\begin{pmatrix}
-8 & 10 & -3 \\
8 & -13 & 6 \\
-3 & 6 & -3
\end{pmatrix}
\]

- **Classical Adjoint** = transpose of the cofactor matrix, \( \text{adj}(A) \)
Matrices

- Inverse = classical adjoint matrix divided by the determinant (HANDBOOK!)

\[
\begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 8
\end{bmatrix}^{-1} = \frac{1}{3} \begin{bmatrix}
-8 & 8 & -3 \\
10 & -13 & 6 \\
-3 & 6 & -3
\end{bmatrix} = \begin{bmatrix}
\frac{-8}{3} & \frac{8}{3} & -1 \\
\frac{3}{10} & \frac{-13}{3} & 2 \\
\frac{3}{3} & \frac{3}{2} & -1
\end{bmatrix}
\]
Vectors

- Scalar, Vector, Tensor
- Unit Vectors \((i, j, k)\)

\[
\vec{A} = a_x \hat{i} + a_y \hat{j} + a_z \hat{k}
\]

\[
\vec{B} = b_x \hat{i} + b_y \hat{j} + b_z \hat{k}
\]

- Vector Operations - Clearly outlined in HANDBOOK!
  - Dot Product, Cross Product
  - Gradient, divergence, and curl (pg. 24)
Example

What is the resultant of the vectors $F_1$, $F_2$, and $F_3$?

$F_1 = 5i + 6j + 3k$  
$F_2 = 11i + 2j + 9k$  
$F_3 = 7i - 6j - 4k$

$\mathbf{R} = (5 + 11 + 7)i + (6 + 2 - 6)j + (3 + 9 - 4)k$

$\mathbf{R} = 23i + 2j + 8k$
Example

What is the angle between the vectors $F_1$ and $F_2$?

$F_1 = 5i + 4j + 6k; \quad F_2 = 4i + 10j + 7k$

$\cos \theta = \frac{F_1 \cdot F_2}{\|F_1\| \|F_2\|}$

$= \frac{(20+40+42)}{\sqrt{25+16+36} \sqrt{16+100+49}} = 0.905$

$\theta = 25.2^\circ$
Example

• For the three vectors \( \mathbf{A} \), \( \mathbf{B} \) and \( \mathbf{C} \), what is the product \( \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) \)?

\[
\begin{align*}
\mathbf{A} &= 6\mathbf{i} + 8\mathbf{j} + 10\mathbf{k} \\
\mathbf{B} &= \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} \\
\mathbf{C} &= 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}
\end{align*}
\]

(A) 0  
(B) 64  
(C) 80  
(D) 216
Differential Calculus

• Derivatives:
  – Definition of a Derivative:
    \[
    y' = \lim_{\Delta x \to 0} \frac{(\Delta y)/(\Delta x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}
    \]
  – Relations among Derivatives (not in handbook):
    \[
    \text{If } x = f(y), \text{ then } \frac{dy}{dx} = 1 + \frac{dx}{dy}
    \]
    \[
    \text{If } x = f(t) \text{ and } y = F(t), \text{ then } \frac{dy}{dx} = \frac{dy}{dt} + \frac{dx}{dt}
    \]
    \[
    \text{If } y = f(u) \text{ and } u = F(x), \text{ then } \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}
    \]
  – Tables of Derivatives
Differential Calculus

• Slope –

The slope of $y(x)$ is $y'(x)$ for all $x$.

(1) Given: $y(x) = 3x^3 - 2x^2 + 7$. What is the slope of the function $y(x)$ at $x = 4$?

(A) 128  (B) 64  (C) 9  (D) 4

$$y'(x) = 9x^2 - 4x$$
$$y'(4) = (9)(4)^2 - (4)(4)$$
$$= 128$$

(2) Given: $y_1' = \frac{1}{2}(1 + 4x - 7 + 2k)$. What is the value of $k$ such that $y_1$ is perpendicular to the curve $y_2 = 2x$ at (1,2)?

(A) 0.25  (B) 0.5  (C) 0.75  (D) 1.0

Perpendicular implies that $m_1m_2 = -1$
Since $y_2'(1) = 2$, then
$$y_1'(1) = -\frac{1}{2} = \frac{1}{2}(1 + (4)(1) - 7 + 2k)$$
$$k = \frac{1}{2}$$
Differential Calculus

- Maxima and Minima

\[ y = f(x) \text{ is maximum for } x = a \text{ where } f'(a) = 0 \text{ and } f''(a) < 0 \]
\[ y = f(x) \text{ is minimum for } x = a \text{ where } f'(a) = 0 \text{ and } f''(a) > 0 \]

What is the maximum of the function \( y = -x^3 + 3x \) for \( x \geq -1 \)?

(A) \(-2\)  (B) \(-1\)  (C) \(0\)  (D) \(2\)

\[ y' = -3x^2 + 3 \]
\[ y'' = -6x \]
When \( y' = 0 = -3x^2 + 3 \)
\[ x^2 = 1; \ x = \pm 1 \]
\[ y''(1) = -6 < 0; \text{ therefore this is a maximum.} \]
\[ y''(-1) = 6 > 0; \text{ therefore this is a minimum.} \]
(Note: Since the minimum is less than the maximum, then all values of \( y \) will be less than the value at the maximum. The point is that local maxima and minima are not necessarily the maximum and minimum of the function. If the problem gives an interval, it’s usually necessary to check the limits of the interval if the problem asks for the maximum or minimum value.)
\[ y(1) = -(1)^3 + 3 = 2 \]
Therefore (D) is correct.
Differential Calculus

• Inflection Points:

\[ y = f(x) \text{ is an inflection point for } x = a \text{ where:} \]
\[ f''(a) = 0 \text{ and} \]
\[ f'''(a) \text{ changes sign about } x = a \]

What is the point of inflection of the function \( y = -x^3 + 3x - 2 \)?
(A) (0,0)  (B) (0,2)  (C) (0,-2)  (D) (2,0)

\[ y' = -3x^2 + 3 \]
\[ y'' = -6x \]
\[ y''' = 0 \text{ when } x = 0 \text{ and } y'' > 0 \text{ for } x < 0; y'' < 0 \text{ for } x > 0 \]
Therefore this is an inflection point.
\[ y(0) = -(0)^3 + 3(0) - 2 = -2 \]
Therefore (C) is correct.
Differential Calculus

- Partial Derivatives:

Treat everything that is not the variable being differentiated as a constant.

What is the partial derivative of $P(R,S,T)$ taken with respect to $T$?

$$P = 2R^3S^2T^{\frac{1}{2}} + R^{\frac{3}{4}}S\cos 2T$$

Everything that is not a function of $T$ is treated as a constant.

$$P = 2R^3S^2(T^{\frac{1}{2}}) + R^{\frac{3}{4}}S(\cos 2T)$$

$$\frac{\partial P}{\partial T} = 2R^3S^2\left(\frac{1}{2}T^{\frac{1}{2}}\right) + R^{\frac{3}{4}}S(-2 \sin 2T)$$

$$= R^3S^2T^{-\frac{1}{2}} - 2R^{\frac{3}{4}}S \sin 2T$$
Differential Calculus

• Curvature (K) of any Curve at P:
  – Rate of change of inclination with respect to its arc length

• Radius of Curvature (R)
  – Radius of a circle that would be tangent to a function at any point

\[
K = \frac{y''}{\left[1 + (y')^2\right]^{\frac{3}{2}}} = \frac{-x''}{\left[1 + (x')^2\right]^{\frac{3}{2}}}
\]

\[
R = \frac{1}{|K|} = \frac{\left[1 + (y')^2\right]^{\frac{3}{2}}}{|y''|} = \frac{\left[1 + (x')^2\right]^{\frac{3}{2}}}{|x'|}
\]
What is the curvature of $y = -x^3 + 3x$ for $x = -1$?

(A) $-2$  
(B) $-1$  
(C) $0$  
(D) $6$

$y' = -3x^2 + 3$ \hspace{1cm} y" = -6x

$y'(-1) = 0$ \hspace{1cm} y"'(-1) = 6

$K = \frac{y''}{3} = \frac{6}{3} = 2

\left[1 + (y')^2\right]^{\frac{3}{2}} \hspace{1cm} \left[1 + (0)^2\right]^{\frac{3}{2}}$

Therefore (D) is correct.
Differential Calculus

• Limits -

If the limit goes to plus or minus infinity, divide the numerator and the denominator by largest power.

Factor out common terms and divide the numerator and the denominator

Looking for \( \lim_{x \to a} \left( \frac{f(x)}{g(x)} \right) \) but \( \left( \frac{f(a)}{g(a)} \right) = \frac{0}{0} \) or \( \left( \frac{f(a)}{g(a)} \right) = \frac{\infty}{\infty} \): use L’Hôpital’s rule.

\[
\lim_{x \to a} \left( \frac{f(x)}{g(x)} \right) = \lim_{x \to a} \left( \frac{f'(x)}{g'(x)} \right) = \lim_{x \to a} \left( \frac{f''(x)}{g''(x)} \right) = \lim_{x \to a} \left( \frac{f'''(x)}{g'''(x)} \right) \ldots
\]

provided the next derivative of \( f(x) \) and \( g(x) \) exist. If not, L’Hôpital’s rule can’t be used.
(1) What is the value of \( \lim_{x \to \infty} \left( \frac{x + 4}{x - 4} \right) \)?

(A) 0 \quad (B) 1 \quad (C) \infty \quad (D) Undefined

Divide the numerator and denominator by \( x \)

\[
\lim_{x \to \infty} \left( \frac{x + 4}{x - 4} \right) = \lim_{x \to \infty} \left( \frac{1 + \frac{4}{x}}{1 - \frac{4}{x}} \right) = \frac{1 + 0}{1 - 0} = 1
\]

Therefore (B) is correct.

What is the limit as \( x \to 2 \) of \( \frac{x^2 - 4}{x - 2} \)?

(A) 0 \quad (B) 2 \quad (C) 4 \quad (D) \infty

Factor out an \((x - 2)\) term in the numerator

\[
\lim_{x \to 2} \left( \frac{x^2 - 4}{x - 2} \right) = \lim_{x \to 2} \left( \frac{(x - 2)(x + 2)}{x - 2} \right) = \lim_{x \to 2} (x + 2) = 2 + 2 = 4
\]

Therefore (C) is correct.

Note: Although L'Hôpital's rule does work on this problem, this solution is faster and easier.
Compute the following limit: $\lim_{x \to 0} \left( \frac{1-\cos x}{x^2} \right)$

(A) 0  (B) $\frac{1}{4}$  (C) $\frac{1}{2}$  (D) $\infty$

Both the numerator and denominator approach 0, so use L’Hôpital’s rule:

$$\lim_{x \to 0} \left( \frac{1-\cos x}{x^2} \right) = \lim_{x \to 0} \left( \frac{\sin x}{2x} \right)$$

Both the numerator and denominator are still approaching 0, so use L’Hôpital’s rule again:

$$\lim_{x \to 0} \left( \frac{\sin x}{2x} \right) = \lim_{x \to 0} \left( \frac{\cos x}{2} \right) = \frac{\cos(0)}{2} = \frac{1}{2}$$

Therefore (C) is correct.
Derivatives of constants are zero, so when the derivative is undone, a constant must be added. To restore the constant, we have to know some value of the original function for some \( x = a \).

What is the constant of integration for \( y(x) = \int (e^{2x} + 2x) \, dx \) if \( y = 1 \) when \( x = 1 \)?

(A) \( C = 2 - e^2 \)  
(B) \( C = -\frac{1}{2} e^2 \)  
(C) \( C = 4 - e^2 \)  
(D) \( C = 1 + 2e^2 \)

\[ y(x) = \frac{1}{2} e^{2x} + x^2 + C \]
\[ y(1) = \frac{1}{2} e^2 + 1 + C = 1 \]
\[ C = -\frac{1}{2} e^2 \]

Therefore (B) is correct.
Integral Calculus

- Indefinite Integrals -

Look for ways to simplify the formula with algebra before integrating. Indefinite integrals can be solved by differentiating the answers, but this is usually the hard way.

Integration by parts

Complicated functions that don’t have integrals listed in the NCEES Handbook can be separated into a product minus an integral that can be integrated with the NCEES Handbook, using the formula for integration by parts:

\[ \int u(x) \, dv(x) = u(x)v(x) - \int v(x) \, du(x) \]

The trick to integration by parts is recognizing the parts that should be \( u(x) \) and \( v(x) \). \( v(x) \) should be a trigonometric function or exponential. \( u(x) \) should be a function that becomes a lower order when differentiated. Usually \( u(x) \) is a polynomial.
Integral Calculus

Find $\int x^2 e^x \, dx$

Let $v(x) = e^x$ and $u(x) = x^2$
so $dv(x) = e^x \, dx$ and $\int x^2 e^x \, dx = x^2 e^x - \int 2xe^x \, dx$

From the NCEES Handbook: $\int xe^{ax} \, dx = \frac{e^{ax}}{a^2} (ax - 1)$

Therefore, $\int x^2 e^x \, dx = x^2 e^x - 2(xe^x - e^x) + C$

Notice that if we had chosen $dv(x) = x^2 \, dx$ and $u(x) = e^x$, the integral gets no better.
Partial fractions

If the integral is the quotient of two complicated polynomials, and the denominator has the higher order, the integral can be simplified by using partial fraction expansion.

Find \[ \int \frac{6x^2 + 9x - 3}{x(x + 3)(x - 1)} \, dx \]
\[
\frac{6x^2 + 9x - 3}{x(x + 3)(x - 1)} = \frac{A}{x} + \frac{B}{x + 3} + \frac{C}{x - 1} = \frac{A(x + 3)(x - 1)}{x(x + 3)(x - 1)} + \frac{B(x)(x - 1)}{x(x + 3)(x - 1)} + \frac{C(x)(x + 3)}{x(x + 3)(x - 1)}
\]

So \(6x^2 + 9x - 3 = A(x + 3)(x - 1) + B(x)(x - 1) + C(x)(x + 3)\)
which gives us three simultaneous equations:

- \(A + B + C = 6\)
- \(2A - B + 3C = 9\)
- \(-3A = -3\)

Therefore \(A = 1, B = 2,\) and \(C = 3\)

\[
\int \frac{6x^2 + 9x - 3}{x(x + 3)(x - 1)} \, dx = \int \frac{1}{x} \, dx + \int \frac{2}{x + 3} \, dx + \int \frac{3}{x - 1} \, dx = \ln|x| + 2 \ln|x + 3| + 3 \ln|x - 1| + C
\]
Integral Calculus

Find the partial fraction expansion of \( \frac{4x - 9}{(x - 3)^2} \)

\[
\frac{4x - 9}{(x - 3)^2} = \frac{A}{x - 3} + \frac{B}{(x - 3)^2} = \frac{A}{x - 3} \frac{(x - 3)}{(x - 3)} + \frac{B}{(x - 3)^2}
\]

so \( 4x - 9 = A(x - 3) + B \)
which gives us two simultaneous equations:
\[
A = 4
\]
\[-9 = -3A + B \]

Thus \( A = 4 \) and \( B = 3 \)

Therefore \( \frac{4x - 9}{(x - 3)^2} = \frac{4}{x - 3} + \frac{3}{(x - 3)^2} \)
Integral Calculus

• Definite Integrals -

Solve the indefinite integral (without the constant of integration).
Evaluate at upper and lower bounds.
Subtract lower bound value from upper bound value.

Find the integral between $\pi/3$ and $\pi/4$ of $f(x) = \sin x$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin x \, dx = \left[-\cos x\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} = -\cos \frac{\pi}{3} - \left(-\cos \frac{\pi}{4}\right)$$

$$= -0.5 + 0.707 = 0.207$$
Integral Calculus

- **Average Value** -

The average value is the definite integral divided by the width of boundary.

\[
\text{Average} = \frac{1}{b-a} \int_{a}^{b} f(x) \, dx
\]

What is the average value of \( y(x) = 2x + 4 \) between \( x = 0 \) and \( x = 4 \)?

(A) 1  (B) 2  (C) 4  (D) 8

\[
\text{Average} = \frac{1}{4-0} \int_{0}^{4} (2x + 4) \, dx = \frac{1}{4} \left[ \frac{2x^2}{2} + 4x \right]_{0}^{4} = \frac{1}{4} \left[ 4^2 + 4(4) \right] = 8
\]
The first definite integral of the difference between two functions is the area between the functions.

$$\text{Area} = \int_{a}^{b} (f_1(x) - f_2(x)) \, dx$$
What is the area between \( y_1 = \frac{1}{4}x + 3 \) and \( y_2 = 6x - 1 \) between \( x = 0 \) and \( x = \frac{1}{2} \)?

(A) 13/35  (B) 11/12  (C) 41/32  (D) 2

\[
\text{Area} = \int_{0}^{\frac{1}{2}} \left( \frac{1}{4}x + 3 \right) - (6x - 1) \, dx = \int_{0}^{\frac{1}{2}} \left( -\frac{23}{4} x + 4 \right) \, dx
\]

\[
= \left[ -\frac{23}{8} x^2 + 4x \right]_{0}^{\frac{1}{2}} = -\frac{23}{8} \left( \frac{1}{2} \right)^2 + \frac{4}{2} = \frac{41}{32}
\]

Integral Calculus
Differential Equations

- **Order of DE** – highest order derivative
- **First-order Homogeneous Equations:**

General form: \( y' + ay = 0 \)

General solution: \( y = Ce^{-ax} \)

Initial condition

Usually \( y(b) = \) constant or \( y'(b) = \) constant

\[ C = \frac{y(b)}{e^{-ab}} \quad \text{or} \quad C = \frac{y'(b)}{e^{-ab}} \]
Differential Equations

Find the solution to the differential equation \( y = 4y' \) if \( y(0) = 1 \).

(A) \( 4e^{-4t} \)  
(B) \( \frac{1}{4}e^{-\frac{1}{4}t} \)  
(C) \( e^{-\frac{1}{4}t} \)  
(D) \( e^{\frac{1}{4}t} \)

Rearranging in standard form
\[ 4y' - y = 0 \]
\[ y' - \frac{1}{4}y = 0 \]

General solution: \( y = Ce^{-at} \)

\[ C = \frac{y(b)}{e^{-ab}} = \frac{y(0)}{\frac{1}{e^4}} = 1 \]

Since \( a = -\frac{1}{4} \) and \( C = 1 \) then \( y = e^{\frac{1}{4}t} \)

Therefore (D) is correct.
Differential Equations

• Separable Equations:

Some differential equations can be separated with all $x$ terms on one side and all $y$ terms on the other. Then the equation can be solved by integrating both sides.

$$ m(x)dx = n(y)dy $$

Reduce $y' + 3(2y - \sin x) - (x(\sin x) + 6y) = 0$ to a separable equation.

$$ y' + 3(2y - \sin x) - (x(\sin x) + 6y) = 0 $$

$$ \frac{dy}{dx} = 3 \sin x + x(\sin x) $$

$$ dy = (3\sin x + x(\sin x))dx $$

Then both sides can be integrated:

$$ y = -3\cos x + (\sin x - x \cos x) + C $$
Differential Equations

• Second-Order Homogeneous Equations:

\[ y'' + 2ay' + by = 0 \]

Characteristic equation: \[ r^2 + 2ar + b = 0 \]
Roots: \[ r = -a \pm \sqrt{a^2 - b} \]

General solutions:
- Overdamped: Real roots \( (a^2 > b) \)
  \[ y = C_1 e^{r_1 x} + C_2 e^{r_2 x} \]
- Critically damped: equal roots \( (a^2 = b) \)
  \[ y = (C_1 + C_2 x) e^{r_1 x} \]
- Underdamped: Complex roots \( (a^2 < b) \)
  \[ y = e^{ax}(C_1 \cos \beta x + C_2 \sin \beta x) \]
  \[ \beta = \sqrt{b - a^2} \]

Initial conditions
- Usually \( y(\text{constant}) = \text{constant} \) and \( y'(\text{constant}) = \text{constant} \)
- Results in two simultaneous equations and two unknowns
Differential Equations

\[ y'' + 6y' + 5y = 0 \]
\[ y(0) = 1 \]
\[ y'(0) = 0 \]

Write the equation in standard form:
\[ y'' + (2)(3)y' + 5y = 0 \]

The characteristic equation is:
\[ r^2 + (2)(3)r + 5 = 0 \]
\[ \text{Roots} = -3 \pm \sqrt{3^2 - 5} = -3 \pm 2 = -1, -5 \]

This is the overdamped case because there are two real roots, so the general solution is:

\[ y = C_1 e^{-1x} + C_2 e^{-5x} \]
\[ y(0) = 1 = C_1 + C_2 \]
\[ y'(0) = 0 = -C_1 - 5C_2 \]
\[ 1 = -4C_2 \]
\[ C_2 = -\frac{1}{4} \]
\[ C_1 = 1\frac{1}{4} \]
\[ y = 1\frac{1}{4}e^{-x} - \frac{1}{4}e^{-5x} \]
Example

The solution to $y'' - 5y' + 6y = 0$ is

a. $y = 5x + c$

b. $y = e^{5x} + c + de^{6x}$

c. $y = Ae^{5x} + Be^{6x}$

d. $y = Ae^{2x} + Be^{3x}$
Probability and Statistics

• Combinations:

\[ C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{r!(n-r)!} \]

A pizza restaurant offers 5 toppings. Given a one-topping minimum, how many combinations are possible?
(A) 5  (B) 10  (C) 31  (D) 36

\[ C_{total} = \sum_{i=1}^{5} C_i = \frac{5!}{1!(5-1)!} + \frac{5!}{2!(5-2)!} + \frac{5!}{3!(5-3)!} + \frac{5!}{4!(5-4)!} + \frac{5!}{5!(5-5)!} \]
\[ = 5 + 10 + 10 + 5 + 1 = 31 \]
Therefore (C) is correct.
Probability and Statistics

• Permutations:

\[ P(n,r) = r! \frac{n!}{(n-r)!} \]

A baseball coach has 11 players on the team. How many possible batting orders are there?

\[ P(11,9) = \frac{11!}{(11-9)!} = 19,958,400 \]
Probability and Statistics

- Laws of Probability:

Probability is defined as a real number between 0 and 1. A probability of 0 is impossible. A probability of 1 is certain.

Probability that either A or B will occur is

\[ P(A + B) = P(A) + P(B) - P(A,B) \]

- \( P(A + B) \) = the probability that either A or B occur alone or that both occur together,
- \( P(A) \) = the probability that A occurs,
- \( P(B) \) = the probability that B occurs, and
- \( P(A, B) \) = the probability that both A and B occur simultaneously.
Probability and Statistics

• Joint Probability:

The probability that both A and B occur is the probability that one will occur times the probability that the other will occur, given the first has occurred.

\[ P(A, B) = P(A) \cdot P(B | A) = P(B) \cdot P(A | B) \]

\[ P(A | B) = \frac{P(A, B)}{P(B)} \]

\[ P(B | A) = \text{the probability that } B \text{ occurs given the fact that } A \text{ has occurred, and} \]

\[ P(A | B) = \text{the probability that } A \text{ occurs given the fact that } B \text{ has occurred.} \]

If either \( P(A) \) or \( P(B) \) is zero, then \( P(A, B) = 0 \).
One bowl contains eight white balls and two red balls. Another bowl contains four yellow balls and six black balls. What is the probability of getting a red ball from the first bowl and a yellow ball from the second bowl on one random draw from each bowl?

(A) 0.08   (B) 0.2   (C) 0.4   (D) 0.8

\[ P(r,y) = P(r)P(y) = (2/10)(4/10) = 0.08 \]

Therefore (A) is correct.

In this example, the success at drawing a red ball does not affect the probability of success at drawing a yellow ball, so \( P(y) = P(y|r) \).
One bowl contains eight white balls, two red balls, four yellow balls, and six black balls. What is the probability of getting a red ball and then a yellow ball drawn at random without replacement?

There are 20 total balls and two are red on the first draw, so $P(r) = 2/20$. Since we assume the first draw was successful, on the second draw there are only 19 balls left and four yellow balls so $P(y|r) = 4/19$. 

$$P(r,y) = P(r) \cdot P(y|r) = (2/20)(4/19) = 8/380 = 0.021$$
Probability and Statistics

• Probability Functions:

Discrete variables have distinct finite number of values (heads or tails, red or blue, 1 or 0, etc.)
Sum total of all outcome probabilities = 1
\[ P(\text{heads}) + P(\text{tails}) + P(\text{edge}) = 1 \]
Note: A “fair coin” is defined as \( P(\text{heads}) = P(\text{tails}) = 0.5 \) and \( P(\text{edge}) = 0 \)
A fair die is defined as \( P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = 1/6. \)

• Binominal Distribution:

All outcomes are either success or failure.
You make \( n \) trials looking for \( x \) successes.
Probability of success = \( p \)
Probability of failure = \( q \)

\[
F(x) = C(n, x) p^x q^{n-x} = \frac{n!}{x!(n-x)!} p^x q^{n-x}
\]
Five percent of students have red hair. If seven students are selected at random, what is the probability that exactly three will have red hair?

\[ F(3) = \frac{7!}{3!(7-3)!} (0.05^3)(0.95^4) = 0.00356 \]
Continuous variables have infinite possible values (3.058261...).
No matter how close two numbers are in value there are always an infinite number of numbers in between.
Continuous variables can have finite or infinite bounds (the probability always \( \to 0 \) as variable \( \to \pm\infty \)).
The sum of all probabilities must be 1 (the area under the curve).

Which of the following cannot be a probability density function?
Probability and Statistics

- **Statistical Treatment of Data:**
  - **Arithmetic Mean:**
    \[ \bar{X} = \mu = \frac{X_1 + X_2 + \ldots + X_n}{n} = \frac{1}{n} \sum_{i=1}^{n} X_i \]
  
  - **Weighted Arithmetic Mean:**
    \[ \bar{X}_w = \frac{\sum w_i X_i}{\sum w_i} \]

<table>
<thead>
<tr>
<th>Data</th>
<th>Weighting factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>62</td>
<td>1</td>
</tr>
<tr>
<td>62</td>
<td>2</td>
</tr>
<tr>
<td>72</td>
<td>3</td>
</tr>
</tbody>
</table>

\[ \bar{X}_w = \frac{\sum w_i X_i}{\sum w_i} = \frac{(1)(62) + (2)(62) + (3)(72)}{1 + 2 + 3} = 67 \]
Probability and Statistics

• Statistical Treatment of Data:
  – Median:

  Half of the data points are less than median; half of the data points are greater than median. The median and mean are equal only for a symmetric distribution.

  What is the median of the following data?
  61, 62, 63, 63, 64, 64, 66, 66, 67, 68, 68, 68, 69, 69, 69, 69, 70, 70, 70, 70, 71, 71, 72, 73, 74, 74, 75, 76, 79

  There are 30 data points, so we start counting at the lowest value and count 15 data points. We see that both the 15th and 16th data points are 69, so the median is 69. If there are an even number of data points and the points on either side of the median are not the same, then the median is half way in between the two middle points.
Probability and Statistics

• Statistical Treatment of Data:
  – Mode: The data value that occurs most frequently

What is the mode of the following data?
61, 62, 63, 63, 64, 64, 66, 66, 67, 68, 68, 69, 69, 69, 69, 70, 70, 70, 70, 71, 71, 72, 73, 74, 74, 75, 76, 79

69 is the most frequently represented number, so it is the mode.

– Variance:

\[
\sigma^2 = \frac{\sum_{i=1}^{N} (X_i - \mu)^2}{N} = \frac{\sum X_i^2}{N} - \mu^2
\]

– Standard Deviation:

\[
\sigma = \sqrt{\frac{\sum_{i=1}^{N} (X_i - \mu)^2}{N}} = \sqrt{\frac{\sum X_i^2}{N} - \mu^2}
\]
Probability and Statistics

- Normal Distribution (Gaussian): averages of $n$ observations tend to become normally distributed as $n$ increases.

$$f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}},$$ where

- $\mu$ = the population mean,
- $\sigma$ = the standard deviation of the population, and
- $-\infty \leq x \leq \infty$
Probability and Statistics

• Normal Distribution (Gaussian): when mean is zero and standard deviation is 1.0 – called standardized or unit normal distribution:

\[ f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \text{ where } -\infty \leq x \leq \infty. \]

• Unit Normal Distribution Table
Probability and Statistics

• Convert distribution to unit normal distribution:

\[ z = \frac{x - \mu}{\sigma} \]

Find the probability on the unit normal chart:

- Column 1: \( f(x) \) = probability of one particular value
- Column 2: \( F(x) \) = probability values \(< x = 1 - R(x)\)
- Column 3: \( R(x) \) = probability values \(> x = 1 - F(x)\)
- Column 4: \( 2R(x) \Rightarrow x + < -x = 1 - W(x)\)
- Column 5: \( W(x) = -x < \text{values} < x = 1 - 2R(x)\)
A normal distribution has a mean of 16 and a standard deviation of 4. What is the probability of values greater than 4?

(A) 0.1295  
(B) 0.9987  
(C) 0.0668  
(D) 0.1336

\[
z = \frac{x - \mu}{\sigma} = \frac{4 - 16}{4} = -3
\]

Due to the symmetry of the distribution, \( R(-z) = F(z) \), so from the NCEES Unit Normal Distribution Table, probability = 0.9987

Therefore (B) is correct.